

On extended umbral calculus, oscillator-like algebras and Generalize Clifford Algebra

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Abstract

Some quantum algebras build from deformed oscillator algebras ala Jordan-Schwinger may be described in terms of a particular case of *psi*-calculus. We give here an example of a specific relation between such certain quantum algebras and generalized Clifford algebras also in the context of Levy -Leblond's azimuthal quantization of angular momentum which was interpreted afterwards as the finite dimensional quantum mechanics by Santhanam et. all. ψ -calculus used for that as a framework is that of classical operator calculus of Rota. By its nature ψ -umbral calculus supplies a simple mathematical underpinning for ψ -deformed quantum-like oscillator algebras and - at least for the $\psi_n(q) = [n_q!]^{-1}$ case [1-3]. It provides the natural underpinning for quantum group investigation. Moreover- the other way around -one may formulate q-extended finite operator calculus with help of the "quantum q-plane" q-commuting variables ... ψ -calculus is expected to be useful in C^* algebraic [4] description of " ψ -quantum processes" with various parastatistics [5].

1 Extended Umbral Calculus in brief

The foundations of what we are going to call " ψ -extension of Rota finite operator calculus" we owe to Viscov [6,7]. ψ -extended umbral calculus is arrived at [6,7,8,9] by considering not only polynomial sequences of binomial type but also of $\{s_n\}_{n \geq 1}$ -binomial type where $\{s_n\}_{n \geq 1}$ -binomial coefficients are defined with help of the generalized factorial $n_s! = s_1 s_2 s_3 \dots s_n$; $S = \{s_n\}_{n \geq 1}$ is an arbitrary F valued sequence with the condition $s_n \neq 0$, $n \in N$. F is a field of char $F = 0$ then the

extensions rely on the notion of ∂_ψ -shift invariance of ∂_ψ -delta operators. Here ∂_ψ denotes the ψ -derivative i.e. $\partial_\psi x^n = n_\psi x^{n-1}$; $n \geq 0$; (then ∂_ψ -linearly extended) and n_ψ denotes the ψ -deformed number where in conformity with Viscov notation $n_\psi \equiv \psi_{n-1}(q)\psi_n^{-1}(q)$; $n_\psi! \equiv \psi_n^{-1}(q) \equiv n_\psi(n-1)_\psi(n-2)_\psi(n-3)_\psi \dots 2_\psi 1_\psi$; $0_\psi! = 1$. We choose to work with \mathfrak{S} - the family of functions sequences such that $\mathfrak{S} = \{\psi; R \supset [a, b]; q \in [a, b]; \psi(q) : Z \rightarrow F; \psi_0(q) = 1; \psi_n(q) \neq 0; \psi_{-n}(q) = 0; n \in N\}$. With the choice $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ we get the well known q-factorial $n_q! = n_q(n-1)_q!; 1_q! = 0_q! = 1$ while the ψ -derivative ∂_ψ becomes now the Jackson's derivative (see: [10-15]) $\partial_q : (\partial_q \varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}$. A polynomial sequence $\{p_n\}_0^\infty$ is then of ψ -binomial type if it satisfies the recurrence

$$E^y(\partial_\psi)p_n(x) \equiv p_n(x + {}_\psi y) = \sum_{k \geq 0} \binom{n}{k}_\psi p_k(x)p_{n-k}(y)$$

where $\binom{n}{k}_\psi \equiv \frac{n_\psi!}{k_\psi!}$. Here $E^y(\partial_\psi) \equiv \exp_\psi\{y\partial_\psi\} = \sum_{k=0}^\infty \frac{y^k \partial_\psi^k}{n_\psi!}$ is a generalized translation operator and ∂_ψ -shift invariance is defined accordingly. Namely we work with \sum_ψ which is the algebra of F -linear operators acting on the algebra P of polynomials. These operators are ∂_ψ -shift invariant operators T i.e. $\forall \alpha \in F; [T; E^\alpha(\partial_\psi)] = 0$. One then introduces the notion of ∂_ψ -delta operator according to

Definition 1.1. Let $Q(\partial_\psi) : P \rightarrow P$; the linear operator $Q(\partial_\psi)$ is a ∂_ψ -delta operator iff a) $Q(\partial_\psi)$ is ∂_ψ -shift invariant; b) $Q(\partial_\psi)(id) = \text{const} \neq 0$.

The strictly related notion is that of the ∂_ψ -basic polynomial sequence:

Definition 1.2. Let $Q(\partial_\psi) : P \rightarrow P$; be the ∂_ψ -delta operator. A polynomial sequence $\{p_n\}_{n \geq 0}$; $\deg p_n = n$ such that: 1) $p_0(x) = 1$; 2) $p_n(0) = 0; n \geq 1$; 3) $Q(\partial_\psi)p_n = n_\psi p_{n-1}$ is called the ∂_ψ -basic polynomial sequence of the ∂_ψ -delta operator $Q(\partial_\psi)$.

After that and among many others the important Theorem (1.1) might be proved using the fact that $\forall Q(\partial_\psi) \exists!$ invertible $S_{\partial_\psi} \in \sum_\psi$ such that $Q(\partial_\psi) = \partial_\psi S_{\partial_\psi}$.

Theorem 1.1 (ψ -Lagrange and ψ -Rodrigues formulas). Let $\{p_n(x)\}_{n=0}^\infty$ be ∂_ψ -basic polynomial sequence of the ∂_ψ -delta operator $Q(\partial_\psi)$: Let $Q(\partial_\psi) = \partial_\psi S$. Then for $n > 0$:

1. $p_n(x) = Q(\partial_\psi)' S^{-n-1} x^n$;
2. $p_n(x) = S^{-n} x^n - \frac{n_\psi}{n} (S_{-n})' x^{n-1}$;
3. $p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi S^{-n} x^{n-1}$
4. $p_n(x) = \frac{n_\psi}{n} \hat{x}_\psi (Q(\partial_\psi)')^{-1} p_{n-1}(x)$ (ψ -Rodrigues formula).

Here we used the properties of the Pincherle ψ -derivative:

Definition 1.3. (compare with (17) in [7]) The Pincherle ψ -derivative i.e. the linear map $' : \sum_{\psi} \rightarrow \sum_{\psi}$.

$$T' = T\hat{x}_{\psi} - \hat{x}_{\psi}T \equiv [T, \hat{x}_{\psi}]$$

where linear operator $\hat{x}_{\psi} : P \rightarrow P$; is defined by

$$\hat{x}_{\psi}x^n = \frac{\psi_{n-1}(q)(n+1)}{\psi_n(q)}x^{n+1} = \frac{n+1}{(n+1)_{\psi}}x^{n+1}; n \geq 0.$$

The more general class constitute Sheffer ∂_{ψ} -polynomials defined as:

Definition 1.4. A polynomial sequence $\{s_n(x)\}_{n=0}^{\infty}$ is called the sequence $\{s_n(x)\}_{n=0}^{\infty}$ of Sheffer ∂_{ψ} -polynomials of the ∂_{ψ} -delta operator iff (1) $s_0(x) = c \neq 0$, (2) $Q(\partial_{\psi})s_n(x) = n_{\psi}s_{n-1}(x)$.

The following proposition relates Sheffer ∂_{ψ} -polynomials of the ∂_{ψ} -delta operator $Q(\partial_{\psi})$ to the unique ∂_{ψ} -basic polynomial sequence of the ∂_{ψ} -delta operator $Q(\partial_{\psi})$:

Proposition 1.1. Let $Q(\partial_{\psi})$ be a ∂_{ψ} -delta operator with ∂_{ψ} -basic polynomial sequence $\{q_n(x)\}_{n=0}^{\infty}$. Then $\{s_n(x)\}_{n=0}^{\infty}$ is a sequence of Sheffer q -polynomials of the ∂_{ψ} -delta operator $Q(\partial_{\psi})$ iff there exists a ∂_{ψ} -shift invariant operator $S_{\partial_{\psi}}$ such that $s_n(x) = S^{-1}q_n(x)$.

The family of Sheffer ∂_{ψ} -polynomials' sequences $\{s_n(x)\}_{n=0}^{\infty}$ corresponding to the fixed ∂_{ψ} -delta operator $Q(\partial_{\psi})$ is labeled by elements of the abelian group of all ∂_{ψ} -shift invariant invertible operators S . It is an orbit of this group.

Examples According to Proposition 1.1 with $Q(\partial_q) = \partial_q$ and $S_{\partial_q} = E^{\alpha}(\partial_q) \exp_{\psi}\{\partial_q^2\}$ we get q -Hermite polynomials while with choice $Q(\partial_q) = \frac{\partial_q}{\partial_q - 1}$ and $S_{\partial_q} = (1 - \partial_q)^{\alpha+1}$ we obtain q -Laguerre polynomials $L_{n,q}^{(\alpha)}(x)$ of order α . So (ψ -extension of finite Rota calculus includes q -Hermite, q -Laguerre polynomials $L_{n,q}^{(\alpha)}(x)$ of order α with their ψ -correspondents. These are already well known q -Sheffer polynomials [16,9]. Specifically q -Laguerre polynomials $L_{n,q}^{(-1)}(x) \equiv L_{n,q}(x)$ form the ∂_q -basic polynomial sequence of $\{L_{n,q}(x)\}_{n \geq 0}$ the ∂_q operator

$$Q(\partial_q) = - \sum_{k=0}^{\infty} \frac{\partial_q^k}{\partial_q - 1} \equiv -[\partial_q + \partial_q^2 + \partial_q^3 + \dots].$$

Using then Theorem (1.1) one arrives at explicit form of $L_{n,q}(x)$. Namely:

$$L_{n,q}(x) = \frac{n_q}{n} \hat{x}_q \left(\frac{1}{\partial_q - 1} \right)^{-n} x^{n-1} = \frac{n_q}{n} \hat{x}_q (\partial_q - 1)^n x^{n-1} = \frac{n_q}{n} \hat{x}_q \sum_{k=1}^n (-1)^k \binom{n}{k}_q \partial_q^{n-k} x^{n-k} = \frac{n_q}{n} \sum_{k=1}^n (-1)^k \binom{n}{k} (n-k)_q^{\frac{n-k}{q}} \frac{k}{k_q} x^k = \frac{n_q}{n} \sum_{k=1}^n (-1)^k \frac{n_q!}{k_q!} \frac{(n-1)_q^{\frac{n-k}{q}}}{(n-k)_q!} \frac{k}{k_q} x^k.$$

So finally

$$L_{n,q}(x) = \frac{n_q}{n} \sum_{k=1}^n (-1)^k \frac{n_q!}{k_q!} \binom{n-1}{k-1}_q \frac{k}{k_q} x^k \quad (1.1)$$

Note: ψ -extended case is covered in this example just by replacement $q \rightarrow \psi$. Let us also stress here again that q -deformed quantum oscillator algebra provides a natural setting for q -Laguerre polynomials and q -Hermite polynomials [40,41]. $sl_q(2)$ and the q -oscillator algebra give rise to basic geometric functions as matrix elements of certain operators in analogy with Lie theory [19]. Also automorphisms of the q -oscillator algebra lead to Sheffer q -polynomials for example to q -generalization of the Charlier polynomials [19].

2 q -extended quantum oscillator and Extended Umbral Calculus

∂_q -delta operators and their duals and similarly ∂_ψ -delta operators with their duals provide us with pairs of generators of ψ -deformed quantum oscillator algebras (see Remark 2.2)- possible candidates to describe parastatistical behavior of some processes [5]. If $\psi_n(q) = [R(q^n)!]^{-1}$ and $R(x) = \frac{1-x}{1-q}$ then we get the well known q -deformed oscillator dual pair of operators leading to the corresponding C^* algebra description of q -Heisenberg-Weyl algebra. These oscillator-like algebras generators and q -oscillator-like algebras generators are encountered explicitly or implicitly in [1,2] and in many other subsequent references - see [30,4,33] and references therein. In many such references [32,33,19] q -Laguerre and q -Hermite or q -Charlier polynomials appear which are just either Sheffer ψ -polynomials or just ∂_ψ -basic polynomial sequences of the ∂_ψ -delta operators $Q(\partial_\psi)$ for $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ and corresponding choice of $Q(\partial_\psi)$ functions of ∂_ψ : $Q = id$. The case $\psi_n(q) = \frac{1}{R(q^n)!}$: $n_\psi = n_k$; $\partial_\psi = \partial_R$ and $n_{\psi(q)} = n_{R(q)} = R(q^n)$ appears implicitly in [21] where advanced theory of general quantum coherent states is being developed. Among others also in [23] it was noticed that commutation relations for the q -oscillator-like algebras generators from [1, 2] and others (see [32, 4]) might be chosen in appropriate operator variables to be of the form [23]:

$$AA^+ - \mu A^+A = 1; \mu = q^2 \quad (2.1)$$

As for the Fock space representation of normalized eigenstates $|n\rangle$ of excitation number operator N various q -deformations of the natural number n are used in literature on quantum groups and at least some families of quantum groups may be constructed from q -analogues of Heisenberg algebra [1,2,23,3,34,35]. In fact, these q -oscillator algebras generators are the so called ∂_q -delta operators $Q(\partial_q)$ i.e. basic objects of the q -extended finite operator calculus of Rota. (Of course $\partial_q \hat{x} - q \hat{x} \partial_q = id$.) The known important fact is that the "q-Canonical Commutation Relations" $AA^+ - qA^+A = 1$ lead [4] to the q -deformed spectrum of excitation number operator N and to various parastatistics [5]. More possibilities result from considerations of Wigner [36] extended by the authors of [5]. We therefore hope that the ψ -calculus of Rota to be developed here might be useful in a C^* algebraic [4] description of " ψ -quantum processes" - if any - with various parastatistics [5]. Here in below via

series of definitions we shall propose a ψ -extension of the q -oscillator model algebra using basic concepts of ψ -extension of calculus of Rota.

Definition 2.1. Let $\{p_n\}_{n \geq 0}$ be the ∂_q -basic polynomial sequence of the ∂_q -delta operator $Q(\partial_q)$. A linear map $\hat{x}_{Q(\partial_\psi)} : P \rightarrow P$; $\hat{x}_{Q(\partial_\psi)} p_n = p_{n+1}$; $n \geq 0$ is called the operator dual to $Q(\partial_q)$.

For $Q = id$ we have: $\hat{x}_{Q(\partial_q)} \equiv \hat{x}_{\partial_q} \equiv \hat{x}$.

Comment: Dual in the above sense corresponds to adjoint in q -umbral calculus language of linear functionals' umbral algebra (see : Proposition 1.1.21 in [16]).

Definition 2.2. Let $\{p_n\}_{n \geq 0}$ be the ∂_ψ -basic polynomial sequence of the ∂_ψ -delta operator $Q(\partial_\psi) = Q$ then the $\hat{q}_{\psi,Q}$ -operator is a liner map; $\hat{q}_{\psi,Q} : P \rightarrow P$; $\hat{q}_{\psi,Q} p_n = \frac{(n+1)_\psi - 1}{n_\psi} p_n$; $n \geq 0$.

We call the $\hat{q}_{\psi,Q}$ operator the $\hat{q}_{\psi,Q}$ -mutator operator.

Note: For $Q = id$, $Q(\partial_\psi) = \partial_\psi$ the natural notation is $\hat{q}_{\psi,id} \equiv \hat{q}_\psi$. For $Q = id$ and $\psi_n(q) = \frac{1}{R(q^n)!}$ and $R(x) + \frac{1-x}{1-q} \hat{q}_{\psi,Q} \equiv \hat{q}_{R,id} \equiv \hat{q}_R \equiv \hat{q}_{q,id} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi,Q} x^n = q^n x^n$.

Definition 2.3. Let A and B be linear operators acting on P ; $A : P \rightarrow P$; $B : P \rightarrow P$. Then $AB - \hat{q}_{\psi,Q} B A \equiv [A, B]_{\hat{q}_{\psi,Q}}$ is called $\hat{q}_{\psi,Q}$ -mutator of A and B operators.

Note: $Q(\partial_\psi) \hat{x}_{Q(\partial_\psi)} - \hat{q}_{\psi,Q} \hat{x}_{Q(\partial_\psi)} Q(\partial_\psi) \equiv [Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = id$. This is easily verified in the ∂_ψ -basic $\{p_n\}_{n \geq 0}$ of the ∂_ψ -delta operator $Q(\partial_\psi)$.

Equipped with pair of operators $(Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)})$ and $\hat{q}_{\psi,Q}$ -mutator we have at our disposal all possible representants of " ψ -canonical pairs" of differential operators on the P algebra. The meaning of the adjective: " ψ -canonical" is explained by the content of the remark 2.2. For important historical reasons however here is at first:

Remark 2.1. The ψ -derivative is a particular example of a linear operator that reduces by one the degree of any polynomial. In 1901 it was proved [26] that every linear operator T mapping P into P may be represented as infinite series in operators \hat{x} and D . In 1986 the authors of [27] supplied the explicit expression for such series in most general case of polynomials in one variable; namely according to the Proposition 1 from [27] one has: "Let Δ be a linear operator that reduces by one each polynomial. Let $\{q_n(\hat{x})\}_{n \geq 0}$ be an arbitrary sequence of polynomials in the operator \hat{x} . Then $T = \sum_{n \geq 0} q_n(\hat{x}) \Delta^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$T = \sum_{n \geq 0} q_n(\hat{x}) \Delta^n \quad (2.2)$$

Note: In 1996 this was extended to algebra of many variables polynomials [28].

Remark 2.2. The importance of the pair of dual operators: $Q(\partial_\psi)$ and $\hat{x}_{Q(\partial_\psi)}$ is reflected by the facts:

- a) $Q(\partial_\psi)\hat{x}_{Q(\partial_\psi)} - \hat{q}_{\psi,Q}\hat{x}_{Q(\partial_\psi)}Q(\partial_\psi) \equiv [Q(\partial_\psi), \hat{x}_{Q(\partial_\psi)}]_{\hat{q}_{\psi,Q}} = id.$
- b) Let $\{q_n(\hat{x}_{Q(\partial_\psi)})\}_{n \geq 0}$ be an arbitrary sequence of polynomials in the operator $\hat{x}_{Q(\partial_\psi)}$. Then $T = \sum_{n \geq 0} q_n(\hat{x}_{Q(\partial_\psi)})Q(\partial_\psi)^n$ defines a linear operator that maps polynomials into polynomials. Conversely, if T is linear operator that maps polynomials into polynomials then there exists a unique expansion of the form

$$T = \sum_{n \geq 0} q_n(\hat{x}_{Q(\partial_\psi)})Q(\partial_\psi)^n \quad (2.3)$$

Equipped with pair of operators $Q(\partial_\psi)$, $\hat{x}_{Q(\partial_\psi)}$ and $\hat{q}_{\psi,Q}$ -mutator we have at our disposal all possible representants of " ψ -canonical pairs" of linear operators on the P algebra such that: a) the above unique expansion $T = \sum_{n \geq 0} q_n(\hat{x}_{Q(\partial_\psi)})Q(\partial_\psi)^n$ holds b) we have the structure of ψ -umbral or ψ -extended finite operator calculus - coworking.

3 No ψ -analogue of quantum q-plane formulation? [37,38]

In [16] Cigler and then Kirichenhofer defined the polynomial sequence $\{p_n\}_0^\infty$ of q-binomial type equivalently by

$$p_n(A+B) \equiv \sum_{k \geq 0} \binom{n}{k}_q p_k(A)p_{n-k}(B) \quad (3.1)$$

where $[B, A]_q \equiv BA - qAB = 0$. A and B might be interpreted then as coordinates on quantum q-plane (see Ref.9 Chapter 4). For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. If so then the following identification holds:

$$p_n(x +_q y) \equiv E_y(\partial_q)p_n(x) = \sum_{k \geq 0} \binom{n}{k}_q p_k(x)p_{n-k}(y) = p_n(\hat{x} + y\hat{Q})\mathbf{1}$$

Also q-Sheffer polynomials $\{s_n(x)\}_{n=0}^\infty$ are defined equivalently (see: 2.1.1. in [13]) by

$$s_n(A+B) \equiv \sum_{k \geq 0} \binom{n}{k}_q s_k(A)p_{n-k}(B) \quad (3.2)$$

where $[B, A]_q \equiv BA - qAB = 0$ and $\{p_n(x)\}_{n=0}^\infty$ of q-binomial type. For example $A = \hat{x}$ and $B = y\hat{Q}$ where $\hat{Q}\varphi(x) = \varphi(qx)$. Then the following identification takes place:

$$s_n(x +_q y) \equiv E^y(\partial_q)s_n(x) = \sum_{k \geq 0} \binom{n}{k}_q s_k(x)p_{n-k}(y) = s_n(\hat{x} + y\hat{Q})\mathbf{1} \quad (3.3)$$

This means that one may formulate q-extended finite operator calculus with help of the "quantum q-plane" q-commuting variables $A, B : AB - qBA \equiv [A, B]_q = 0$. One may now be tempted - perhaps in vain - to formulate the basic notions of ψ -extended finite operator calculus with help of the "quantum ψ -plane" $\hat{q}_{\psi, Q}$ -commuting variables $A, B : [A, B]_{\hat{q}_{\psi, Q}} = 0$ exactly in the same way as in [16]. For that to try consider appropriate generalization of $A = \hat{x}$ and $B = y\hat{Q}$ where this time the action of \hat{Q} on $\{x^n\}_0^\infty$ is to be found from the condition $AB - \hat{q}_{\psi}BA \equiv [A, B]_{\hat{q}_{\psi}} = 0$. Acting with $[A, B]_{\hat{q}_{\psi}}$ on $\{x^n\}_0^\infty$ due to $\hat{q}_{\psi}x^n = \frac{(n+1)\psi-1}{n_{\psi}}x^n$; $n \geq 0$ one easily sees that now $\hat{Q}x^n = b_n x^n$ where $b_0 = 0$ and $b_n = \prod_{k=1}^n \frac{(k+1)\psi-1}{k_{\psi}}$ for $n > 0$ is the solution of the difference equation: $b_n - bn - 1 \frac{(n+1)\psi-1}{n_{\psi}} = 0$; $n > 0$. With all above taken into account one immediately verifies that for our A and B \hat{q}_{ψ} -commuting variables already

$$(A + B)^n \neq \sum_{k \geq 0} \binom{n}{k}_{\psi} A^k B^{n-k} \quad (3.4)$$

unless $\psi_n(q) = \frac{1}{R(q^n)!}$; $R(x) = \frac{1-x}{1-q}$ hence $\hat{q}_{\psi, Q} \equiv \hat{q}_{R, id} \equiv \hat{q}_R \equiv \hat{q}_{q, id} \equiv \hat{q}_q \equiv \hat{q}$ and $\hat{q}_{\psi, Q}x^n = q^n x^n$. Concluding: the above identifications of polynomial sequence $\{p_n\}_0^\infty$ of q-binomial type and Sheffer q-polynomials $\{s_n(x)\}_{n=0}^\infty$ fail to be extended to the more general ψ -case. This means that we can not formulate that way the ψ -extended finite operator calculus with help of the "quantum ψ -plane" $\hat{q}_{\psi, Q}$ -commuting variables $A, B : AB - \hat{q}_{\psi, Q}BA \equiv [A, B]_{\hat{q}_{\psi, Q}} = 0$ while considering algebra of polynomials P over the field F .

4 Relation to Quantum groups - Polar decomposition of $SU_q(2; C)$ group algebra

The standard basis of Lie algebra $su(2) = so(3)$ is well known to be represented by:

$$\begin{aligned} J_3 &= \sum_{m=-j}^j m |jm\rangle; \\ J_+ &= \sum_{m=-j}^j \sqrt{(j-m)(j+m+1)} |j(m+1)\rangle \langle jm|; \\ J_- &= \sum_{m=-j}^j \sqrt{(j+m)(j-m+1)} |j(m+1)\rangle \langle jm| \end{aligned} \quad (4.1)$$

In [1] Biedenharn proposed a new realization of quantum group $SU_q(2)$ and in order to realize generators of a q-deformation $U_q(su(2))$ of the universal enveloping algebra of the Lie algebra $su(2)$ he defined a pair of mutual commuting q-harmonic oscillator systems (à la Jordan-Schwinger approach to $su(2)$ Lie algebra).

At the same time [2] Mac Farlane had also discovered q-oscillator description of $SU_q(2)$ (alà Jordan-Schwinger approach to $su(2)$ Lie algebra). The generators of a q-deformation $U_q(su(2))$ of the universal enveloping algebra of the Lie algebra $su(2)$ (called by physicists "generators of the quantum group $SU_q(2)$ " - which is not a group!) are obtained from (4.1) by one of several possible q-deformations. In Biedenharn's and Mac Farlane's case one uses the following q-deformation of numbers and operators:

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (4.2)$$

Thus q-deformed 4.1 representation of generators now reads

$$\begin{aligned} J_3 &= \sum_{m=-j}^j m |j, m\rangle_q; \\ J_+ &= \sum_{m=-j}^j \sqrt{[j-m]_q [j+m+1]_q} |j, (m+1)\rangle_{qq} \langle j, m|; \\ J_- &= \sum_{m=-j}^j \sqrt{[j+m]_q [j-m+1]_q} |j, (m+1)\rangle_{qq} \langle j, m|, \end{aligned} \quad (4.3)$$

where

$$|j, m\rangle_q = |j+m\rangle_q |j-m\rangle_q = \frac{a_{1q}^{+j+m} a_{2q}^{+j-m}}{[j+m]_q! [j-m]_q!} |0\rangle_q \quad (4.4)$$

and a_{1q}^+, a_{2q}^+ represent two mutually commuting creation operators of q-quantum oscillators. Corresponding commutation relations of the generators of a q-deformation $U_q(su(2))$ of the universal enveloping algebra of the Lie algebra $su(2)$ are of the familiar though now q-deformed form [1],[2],[29]:

$$[J_3, J_+] = J_+; [J_3, J_-] = -J_-; [J_+, J_-] = [2J_3]_q. \quad (4.5)$$

From (4.3) one may derive [29] the polar decomposition of the generators J_+, J_- :

$$J_+ = \sqrt{J_+ J_-} \sigma_1^{-1} = \sigma_1^{-1} \sqrt{J_- J_+} \quad J_- = \sqrt{J_+ J_-} \sigma_1 = \sigma_1 \sqrt{J_- J_+}, \quad (4.6)$$

where σ_1 is the first of the two generators of generalized Pauli algebra [30]

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = (n \times n) \quad (4.7)$$

The second generator σ_2 has been also used in [29] in order to remark on relevance of such a pair σ_1, σ_2 to $GL_\omega(2; C)$ properties ($q \equiv \omega \equiv \exp\{\frac{2\pi i}{n}\}$). It is to be noted here that the polar decomposition for undeformed $su(2) = so(3)$ algebra of undeformed quantum angular momentum had been performed already by Levy-Leblond in [31]. There he had interpreted such a polar decomposition as the "azimuthal quantization of angular momentum". Following the authors of [29] it should be noted here that *generalized Pauli algebra appears in q-deformed and in undeformed case of polar decomposition in the same way (4.6)*. Neither Levy-Leblond nor the authors of [29] had realized that they are dealing with generalized Pauli algebra [30]. These has been realized afterwards by T.S. Santhanam [32]. In the notation of [30] and papers quoted there

$$\sigma_2 \equiv U = \omega^Q = \exp\left\{\frac{2\pi i}{n}Q\right\} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \omega & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \omega^{n-1} \end{pmatrix} \quad (4.8)$$

where

$$Q = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & n-1 \end{pmatrix} \quad (4.9)$$

$$\sigma_1 \equiv V = \omega^P = \exp\left\{\frac{2\pi i}{n}P\right\} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \quad (4.10)$$

where

$$P = S^\dagger Q S = (P_{\alpha, \kappa}), \quad (4.11)$$

$$P_{\alpha, \kappa} = \begin{cases} 0 & \alpha = \kappa \\ [\bar{\omega}^{\alpha - \kappa} - 1]^{-1} & \alpha \neq \kappa \end{cases}$$

and

$$s = (\langle \tilde{k} | l \rangle) = \frac{1}{\sqrt{n}} (\omega^{kl})_{k, l \in Z_n} \quad (4.12)$$

is the Sylvester matrix. Formulas (4.8)-(4.12) contain the main information on quantum kinematics of the finite dimensional quantum mechanics as here we interpret polar decomposition of quantum angular momentum algebra $su(2) = so(3)$ formalism as a model of finite dimensional quantum mechanics with the classical phase space being the torus $\mathbf{Z}_n \times \mathbf{Z}_n$ (see [30] and references therein). This possibility was already considered by Weyl in [33]. "Azimuthal quantization of angular momentum" was interpreted afterwards as the finite dimensional quantum mechanics by Santhanam et. all [32]. The considerations of this section allow us to hope to elaborate soon more on the q-deformed finite dimensional quantum mechanics treated as an interpretation of q-deformed $su(2)$ algebra of q-deformed

angular momentum [34,35]. A specific relation between certain quantum algebras and generalized Clifford algebras was recently discovered in [36]. More elaborated presentation of the content of Sections I, II and III one may find in [37,38].

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